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UNIQUE PRIMITIVE PYTHAGOREAN TRIPLES FOR EVERY INTEGER, FOR EVERY SET OF TWO INTEGERS, AND FOR EVERY SET OF THREE INTEGERS, ONE OF WHICH IS A PRIME

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Abstract

Euclid presents formulae to generate primitive Pythagorean triples. The formula given by Euclid requires an input of two parameters, herein called M and N. Euclid's formula has the "advantages" of (1) being unique, meaning any change in either M or N will change the triple that results; and (2) being exhaustive, meaning all primitive Pythagorean triples may be generated by Euclid's formula. However, it has the "disadvantage" that there is a dependency between M and N, and they cannot be chosen independently. Davis, Appel and Seff (2019) (henceforth DAS) previously presented two alternate formulae to generate primitive Pythagorean triples that do not have this disadvantage. The first formula only requires one parameter which may be any positive integer, and the second requires an input of two parameters that are totally independent, and may be any

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positive integers. These two formulae are also unique, but they are not exhaustive—there exist primitive Pythagorean triples that are not generated by either the first or second of these formulae. In this paper we extend their work by presenting a third formula which requires an input of three parameters that are totally independent, and may be any positive integers, except that one of the parameters must be a prime number.

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1. Pythagorean Triples: Notation, Definitions, Basic Facts

1.1 Notation: Throughout this paper, all variables refer to positive integers.

1.2 Definitions and Basic Facts:

a. A *Pythagorean triple* is a set of any three positive integers, (A, B, C), such that $A^2 + B^2 = C^2$. (Note by requiring A and B to be positive, we exclude the trivial case when A or B = 0.) Famous examples are (3, 4, 5), (5, 12, 13), and multiples of them, such as (6, 8, 10) or (30, 40, 50).

b. A Pythagorean triple is said to be *primitive* if there is no positive integer greater than one that divides each of A, B, or C. Since a divisor of any two of the equation X + Y = Z divides the third, the condition of dividing just two of the three of A, B, C is equivalent to dividing all three, so it follows that if any number divides just two of the three A, B, C, then the triple is not primitive.

2. Euclid's Formula for Primitive Pythagorean triples

(A, B, C) is a primitive Pythagorean triple if and only if there exist two positive integers, M and N, satisfying four conditions:

(1) N > 0

(2) M > N

(3) M and N are of opposite parity (one is odd and the other is even),

(4) M and N are relatively prime

where

$A = M^2 - N^2$

$B = 2MN$

$C = M^2 + N^2$

These last three equations are called "Euclid's formula for primitive Pythagorean triples[i]."

In the general solution, M and N **cannot** be chosen independently, since M and N depend on each other. For instance, if M is odd, then N must be even.

In this paper we are looking for solutions in which we can choose **any** one integer as an input parameter, or **any** set of two integers as input parameters, or **any** set of three integers (one of which is a prime) to generate a primitive Pythagorean triple.

2.1 First Formula

For the first formula, DAS set[ii]:

$N(I) = I$

$M(I) = N + 1$

where I is **any** positive integer.

For this formula,

$A(I) = 2I + 1$

$B(I) = 2I^2 + 2I$

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$C(I) = 2I^2 + 2I + 1$

Furthermore,

$C – A = 2N^2 = 2I^2$

$C + A = 2M^2 = 2(I + 1)^2$

$(C – B)^{1/2} = M - N = 1$

$(C + B)^{1/2} = M + N = 2I + 1$

2.2 Second Formula

For the second formula, DAS set:

$N(I) = 2^{I-1}$

$M(I, J) = N + 2J +1$

where I and J are **any** two positive integers.

For this equation,

$A(I, J) = (2^{I-1} + 2J + 1)^2 – 2^{2I-2}$

$B(I, J) = 2^I (2^{I-1} + 2J + 1)$

$C(I, J) = (2^{I-1} + 2J + 1)^2 + 2^{2I-2}$

Furthermore,

$C – A = 2N^2 = 2^{2I-1}$

$C + A = 2M^2 = 2(2^{I-1} + 2J +1)^2$

$(C – B)^{1/2} = M - N = 2J + 1$

$(C + B)^{1/2} = M + N = 2^I + 2J + 1$

2.3 Third Formula

In this paper we introduce a third formula, which is a generalization of the second formula. In the second formula N is a power of 2, and in this new third formula N is a power of an arbitrary prime, P. Thus, this third formula will generate a primitive Pythagorean triple given **any** input set of three positive integers, two of them (I and J) are arbitrary, and the third, P, is an arbitrary prime.

We set:

$N(I, P) = P^I$

$M(I, J, P) = N + 2JP - 1$

where I and J are **any** two positive integers, and P is **any prime** number.

The following is a proof that these equations satisfy the four conditions of Euclid's formula, and they thus produce unique primitive Pythagorean triples:

Condition 1: Since P and I are positive, $N = P^I$ is positive.

Condition 2: Since P is a prime and therefore ≥ 2, it follows that $2JP - 1 \geq 3$, and therefore positive. Hence $M > N$, since M = N + a positive number.

Condition 3: Since M =N + an odd number, M and N are of opposite parity.

Condition 4: P is the only prime dividing N, and when M is divided by P, the remainder is $P - 1$, it follows that M and N are relatively prime.

Therefore, this formula produces primitive Pythagorean triples which are all unique.

For this equation,

$A(I, J, P) = (P^I + 2JP - 1)^2 - P^{2I}$

$B(I, J, P) = 2P^I (P^I + 2JP - 1)$

$C(I, J, P) = (P^I + 2JP - 1)^2 + P^{2I}$

Furthermore,

$C - A = 2N^2 = 2P^{2I}$

$C + A = 2M^2 = 2(P^I + 2JP - 1)^2$

$(C - B)^{1/2} = M - N = 2JP - 1$

$(C + B)^{1/2} = M + N = 2P^I + 2JP - 1$

Table 1 presents a summary of the three equations and their relationship to Euclid's general solution.

References

Harry Z. Davis, Solomon Appel, David Seff (2019), Unique Primitive Pythagorean Triples for Every Integer and for Every Set of Two Integers, International Journals of Multidisciplinary Research Academy

[i]There are formulae other than Euclid's that produce all primitive Pythagorean triples. For alternative formulations, see Wikipedia, "Pythagorean triple" and Douglas W. Mitchell "85.27 An Alternative Characterisation of All Primitive Pythagorean Triples." *The Mathematical Gazette*, vol. 85, no. 503, 2001, pp. 273–275. *JSTOR*, JSTOR, www.jstor.org/stable/3622017.
[ii] DAS (2019) prove that both their equations satisfy the four conditions of Euclid's formula, and they thus produce unique primitive Pythagorean triples.

	General Solution	Formula I	Formula II	Formula III
Independent Variable(s)		I = any integer	I, J = any integer	I, J = any integer P = any prime
N		N(I) = I	$N(I) = 2^{I-1}$	$N(I, P) = P^I$
M		M(I) = N + 1	M(I, J) = N + 2J + 1	M(I, J, P) = N + 2JP - 1
Four Conditions 1) N> 0		N = I, I>0	$N = 2^{I-1}$ I>0	$N = P^I$ I> 0, P≥ 2
2) M > N		M = N + 1	M = N + (2J + 1) J > 0	M = N + (2JP – 1) J>0, P ≥ 2
3) M, N Opposite Parity		M = N + 1 1 = odd	M = N + (2J + 1) 2J + 1 =odd	M = N + (2JP – 1) 2JP – 1 = odd
4) M, N Relatively Prime		M = N + 1	$N = 2^{I-1}$, only factor = 2 $M = 2\,(2^{I-2} + J) + 1$	$N = P^I$, only factor = P $M = P\,(P^{I-1} + J) - 1$
A	$M^2 - N^2$	A(I) = 2I + 1	$A(I, J) = (2^{I-1}+ 2J +1)^2 – 2^{2I-2}$	$A(I, J, P) = (P^I+ 2JP - 1)^2 – P^{2I}$
B	2MN	$B(I) = 2I^2 + 2I$	$B(I, J) = 2^I\,(2^{I-1}+ 2J + 1)$	$B(I, J, P) = 2P^I\,(P^I+2JP - 1)$
C	$M^2 + N^2$	$C(I) = 2I^2 + 2I + 1$	$C(I, J) = (2^{I-1}+ 2J + 1)^2 + 2^{2I-2}$	$C(I, J, P) = (P^I+ 2JP - 1)^2 + P^{2I}$
C- A	$2N^2$	$2I^2$	2^{2I-1}	$2P^{2I}$
C + A	$2M^2$	$2(I + 1)^2$	$2(2^{I-1}+ 2J +1)^2$	$2(P^I+ 2JP - 1)^2$
$(C-B)^{1/2}$	M – N	1	2J + 1	2JP - 1
$(C+B)^{1/2}$	M + N	2I + 1	$2^I + 2J + 1$	$2P^I + 2JP - 1$